Kirigami

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1 Introduction

Kirigami is a variation on origami, the art of paper folding. Where origami is named from the Japanese ori meaning "folding", and kami meaning "paper", kiri means "cut". Kirigami thus allows cutting of the paper, along with folding and creasing.

Kirigami, as an art form, has much more freedom in structure than origami. As origami is traditionally folded from a single sheet, the ability to reshape and restructure a sheet through cutting allows for a plethora of new orientations. A simple and common example of kirigami is a pop-up card, transforming the two dimensional card into a pleasing three dimensional visual through a single unfolding motion (Figure 1). The artist Ingrid Siliakus, who is known as a "paper architect", utilizes kirigami concepts to create massive, intricate sculptures, transforming flat surfaces into eye-poppingly complicated three dimensional structures (Figure 2).

Figure 1: A pop-up kirigami card

Figure 2: Sculpture by Siliakus

Outside of their artistic merit and aesthetic appeal, origami/kirigami concepts are now being explored by scientists and mathematicians for more practical purposes, such as engineering 3D structures. This has led to the development of the field of computational origami, or the the algorithmic aspects of paper folding. This paper will explore some of the mathematics behind paper folding and cutting, specifically focusing on the concepts of surface curvature.

2 Motivation

Origami and kirigami are exciting concepts. They start with flat, 2D surfaces, and result in complex 3D geometries, all without changing the intrinsic geometry of the original material. It preserves the surface features, such as curvature while simultaneously allowing the manipulation of the surface into a shape that appears to have curvature. In engineering, this leads to prospects that include foldable solar panels, collapsible robots, and modular construction pieces. It is far easier and cheaper to mass-produce a flat, nearly two dimensional object can that then change its orientation to fit particular needs. Other interests, like artificial membranes and batteries, might require the preservation of structures on or beneath the surface, something that origami and kirigami approaches naturally take into account. Overall, there are lots of potential improvements and discoveries to be made in many different fields.

3 Curvature

Let us start by introducing the idea of curvature. Loosely, curvature is the idea of how close something is to being "straight" or "flat", adjectives that change depending on context and dimension. For a smooth curve, curvature is defined by the speed at which that curve is changing direction. In relation to origami and kirigami, we are most interested in curvature on a 2D surface in $\mathbb{R}^3$. At some point on a surface, we can find a normal vector at that point. By slicing planes that contain that normal vector through the point, we produce a series of 2D curves. These curves have varying curvatures, and the maximum and minimum values of these curvatures are called the principal curvatures. The Gaussian curvature κ is simply the product of the two principal curvatures. Intuitively, this gives us a few ideas on how Gaussian curvature works: it can only be 0 if one of the two principal curvatures is 0, and increases or decreases depending on the principal curvatures also. An alternative definition of Gaussian curvature, the spherical representation, is as follows. For point P on an arbitrary surface, we take the set of normal unit vectors on the surface at the boundary of a closed path that contains P. The area of this boundary is called Γ. Then we translates the vectors to the center of a unit sphere. The vectors then will trace out another closed boundary on this unit sphere, the area of which we call Γ'. This is called the Gauss map (Figure 3), and we define our Gaussian curvature as

$$\kappa = \lim_{\Gamma \to P} \frac{\Gamma}{\Gamma'}$$

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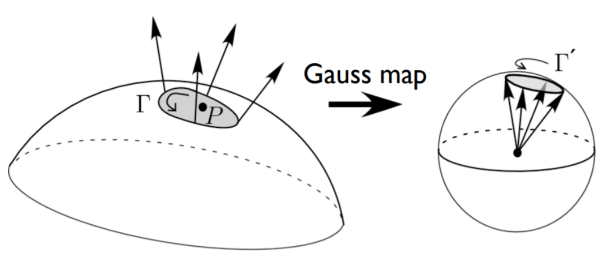


Figure 3: The Gauss Map and Gaussian Curvature

Mathematically, we can think about a piece of paper as a plane. The plane can have some arbitrary length and width, and like an ideal piece of paper, no thickness. Using the spherical definition of Gaussian curvature for intuition, a flat piece of paper should have no curvature, regardless of how it is bent, because the area of the boundary produced by the Gauss map will never enclose any area. For instance, an untouched flat piece of paper produces a point after applying the Gauss map, which certainly has no area, so the Gaussian curvature is 0. Similarly, for a bent or curved sheet of paper, the Gauss map traces out a line, but still encloses no area, also resulting in a 0 curvature. In fact, for origami, we can calculate whether certain folds are rigid foldable by calculating the Gaussian curvature, since it should always be 0 for rigid folds. Given this, it might seem odd to use kirigami or origami in these situations, since because the curvature is 0 for these rigid folds, it will be impossible to create shapes that have true curvature, such as a sphere or a saddle. However, the strength of these techniques in areas like engineering comes from the fact that they can be used to approximate and simulate curvature to possibly achieve the best of both worlds.

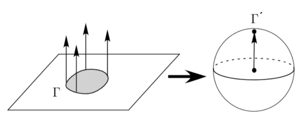
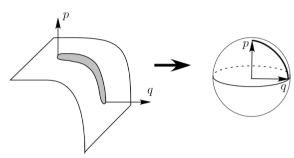


Figure 4:



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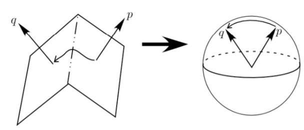

Figure 6:

4 Kirigami and curvature

The Miura-ori origami fold, which we saw in class, is a way of folding a sheet or plane into a smaller area. It has one degree of freedom, and only allows "furling" and "unfurling". However, it is interesting because with paper, which can bend, twisting and deforming a non-rigid Miura-ori fold, allowing bending of its faces, produces shapes that seemingly have curvature (Figure 9)! However, while this is okay at a glance for simulating simple curves like a saddle or a sphere, this has difficulty simulating more complex shapes with bigger jumps in curvature or more rugged contours. Kirigami approaches are much more robust and do not require warping of the faces locally.

We now explore the lattice kirigami method, which involves removing area from a 2D tesselation of regular hexagons. The key idea is that pasting edges together from removed areas results in a 3D "stepped" surface with changing Gaussian curvature from area to area, but 0 net curvature (no warping or deformation of the surface). For instance, removing a $\pi/6$ wedge from a hexagon in the tesselation and extending it to the centers of two neighboring hexagons and pasting edges together produces a pentagon and two partial hexagons into a heptagon. This is known as a "5-7" dipole (Figure 8). Similarly, cutting out a $\pi/3$ wedge can produce a "2-4" dipole (Figure 9).

Cutting out an entire hexagon can produce a "sixon". All of these cuts allow for "popping" the height of the paper up or down and producing a local change

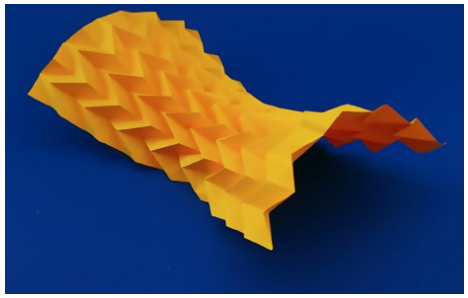

Figure 7: A non-rigid Miura-ori saddle

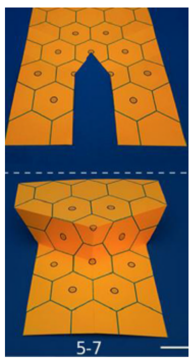

Figure 8:

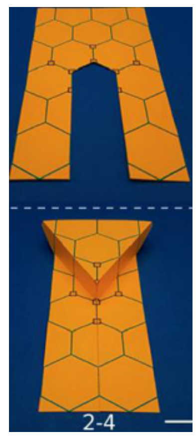

Figure 9:

in elevation. With a tessellation of sixons, Sussman et al showed that one can arbitrarily simulate curvature of some shape by using these kirigami techniques and putting together step-wise plateaus to approximate a surface gradient. It also completely preserves the properties of the flat plane, where locally all net curvature is 0, and is relatively simple compared to some origami techniques (Callens). Sussman demonstrates a practical application of approximating such a surface, creating a kirigami structure of the contours of Mt. Katahdin (Figure 11).

5 Conclusion

The science behind kirigami and origami contain powerful techniques suited toward a variety of uses, from entertainment to building robots to architecture. In particular, we saw that though paper has no curvature, cutting and folding techniques can be used to simulate curvature on a larger scale, and kirigami cutting techniques can even be used to simulate almost any arbitrary curvature

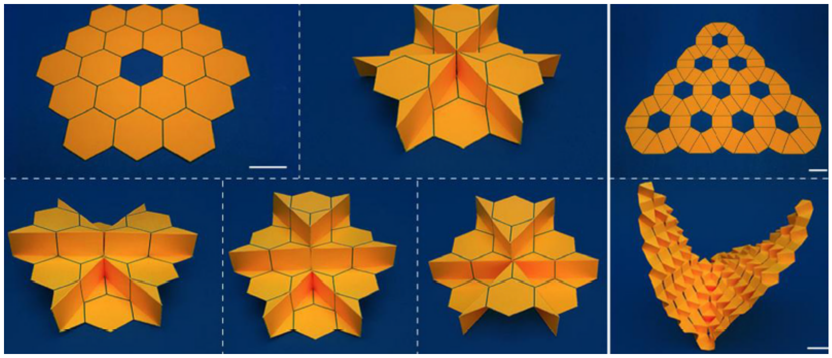

Figure 10: a demonstration of sixons

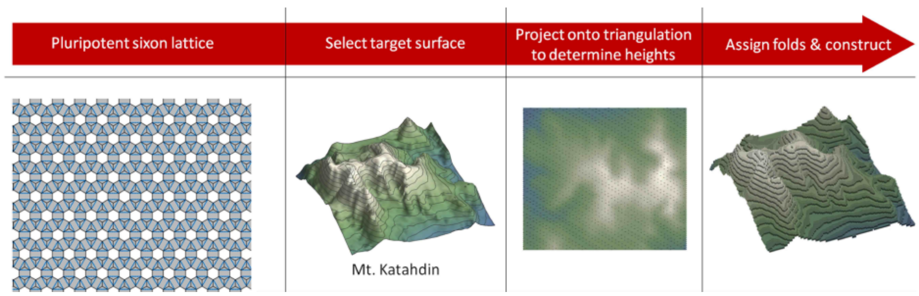

Figure 11: Practical applications of sixons

to a relatively high level of accuracy.

6 References

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